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Demonstrations of geometric algebra for mathematicians

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§ Introduction. The notion of the Clifford algebra is widely known among mathematicians. But it is hardly known (particularly in Japan) that recently some people study this algebra with great enthusiasm, and a large number of applications to physics and engineering have been developed. The members of the research groups call their own object *geometric algebra* rather than Clifford algebra because of its "geometric nature", returning the original terminology by Clifford himself. One of their key concepts is to consider the Grassmann algebra inside the Clifford algebra. In this article, we present briefly what we can do with this idea in elementary linear algebra. Our main reference is [2, Section 1].

§1. Preliminary. Let V be an n-dimensional Euclidean space, and $C(V)$ the Clifford algebra of V. Namely, $C(V)$ is the associative algebra generated by V together with the real number field $\mathbb{R}$ with a relation $x^2 = \|x\|^2$ $(x \in V)$. Then $(xy + yx)/2$ equals the scalar product $(x|y)$ for vectors $x, y \in V$. Thus, if x and y are orthogonal, we have $xy = -yx$.

For $v_1, v_2, \dots, v_r \in V$, we define their *outer product* by

$$(1) \qquad v_1 \wedge v_2 \wedge \cdots \wedge v_r := \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} (\operatorname{sgn}\sigma) v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(r)} \in C(V)$$

and call such an element of $C(V)$ a *simple r-vector* or an *r-blade*. A linear combination of simple r-vectors is called an r-*vector*, and the set of r-vectors is denoted by $C^r(V)$. Like the ordinary exterior product of the Grassmann algebra, the outer product is anti-commutative. If the vectors $v_1, \dots, v_r$ are mutually orthogonal, the outer product equals the Clifford product: $v_1 \wedge v_2 \wedge \cdots \wedge v_r = v_1 v_2 \cdots v_r$ because $(\operatorname{sgn}\sigma) v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(r)} = v_1 v_2 \cdots v_r$ for each $\sigma \in \mathfrak{S}_r$ in this case.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V. For a set $I = \{i_1, i_2, \dots, i_r\}$ $(1 \le i_1 < i_2 < \cdots < i_r \le n)$, we write e_I for the product $e_{i_1} e_{i_2} \cdots e_{i_n} \in C(V)$, and for

$I = \emptyset$ we put $e_I := 1 \in C(V)$. Then $\{e_I\}_{I \subset \{1,\dots,n\}}$ is a basis of $C(V)$ as a linear space. Since e_I is equal to the outer product $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$, we have $C^r(V) = \sum_{\sharp I = r}^{\oplus} \mathbb{R} e_I$, so that we get a direct sum decomposition of $C(V)$ as a linear space:

$$C(V) = \mathbb{R} \oplus V \oplus C^2(V) \oplus \cdots \oplus C^n(V). \tag{2}$$

Let $\dagger : C(V) \to C(V)$ be the canonical anti-automorphism ("*reverse*"), that is, the linear map on $C(V)$ with the properties

$$1^\dagger = 1, \quad v^\dagger = v \ (v \in V), \quad (AB)^\dagger = B^\dagger A^\dagger \ (A, B \in C(V)).$$

For elements A, B of $C(V)$, we denote by $(A|B)$ the scalar part of $AB^\dagger \in C(V)$ with respect to the decomposition (2). If $A = \sum_I a_I e_I$ and $B = \sum_I b_I e_I$, we have $(A|B) = \sum_I a_I b_I$, so that $(\cdot|\cdot)$ defines a positive definite scalar product on $C(V)$ extending the original scalar product on V. For $A \in C(V)$, we write $\|A\|$ for the Euclidean norm $(A|A)^{1/2}$ of A.

§2. Inverse of a simple r vector. In what follows, we assume that vectors $v_1, v_2, \dots, v_r \in V$ are linearly independent, which is equivalent to $v_1 \wedge v_2 \wedge \cdots \wedge v_r \neq 0$.

Proposition 1. (i) *The norm $\|v_1 \wedge v_2 \wedge \cdots \wedge v_r\|$ gives the volume of the parallelotope spanned by $v_1, v_2, \dots, v_r$.*
(ii) *The element $A := v_1 \wedge v_2 \wedge \cdots \wedge v_r$ is invertible as an element of $C(V)$. Indeed, one has*

$$A^{-1} = \frac{1}{\|A\|^2} A^\dagger = \frac{1}{\|v_1 \wedge v_2 \wedge \cdots \wedge v_r\|^2} v_r \wedge \cdots \wedge v_2 \wedge v_1.$$

For the proof of Proposition 1, we state the following lemma, which is equivalent to the validity of the Gram-Schmidt orthogonalization (see Proposition 3 (ii)).

Lemma 2. *For linearly independent vectors $v_1, v_2, \dots, v_r$, we can take a unique family of coefficients $c_{ij} \in \mathbb{R}$ $(i < j)$ for which the vectors*

$$u_1 := v_1, \qquad u_j := v_j + \sum_{i=1}^{j-1} c_{ij} v_i \quad (j = 2, \dots, r) \tag{3}$$

are mutually orthogonal.

Now let us prove Proposition 1. From (3) and the orthogonality of $u_1, \dots, u_r$, we see that

$$A = v_1 \wedge v_2 \wedge \cdots \wedge v_r = u_1 \wedge u_2 \wedge \cdots \wedge u_r = u_1 u_2 \cdots u_r. \tag{4}$$

Then we have

$$AA^\dagger = (u_1 u_2 \cdots u_r)(u_r \cdots u_2 u_1) = \|u_1\|^2 \|u_2\|^2 \cdots \|u_r\|^2.$$

Since $AA^\dagger$ belongs to $\mathbb{R}$, we have $AA^\dagger = \|A\|^2$. Thus $\|A\| = \|u_1\|\|u_2\| \cdots \|u_r\|$, which implies the assertion (i). On the other hand, we observe

$$A^\dagger A = (u_r \cdots u_2 u_1)(u_1 u_2 \cdots u_r) = \|u_1\|^2 \|u_2\|^2 \cdots \|u_r\|^2 = \|A\|^2.$$

Therefore $A(\frac{1}{\|A\|^2} A^\dagger) = (\frac{1}{\|A\|^2} A^\dagger)A = 1$, which means the assertion (ii). □

As we shall see below, it is a great advantage over the ordinary Grassmann algebra that we can take the inverse of the simple r-vector $v_1 \wedge \cdots \wedge v_r$ in the Clifford algebra. The argument in the proof of Proposition 1 tells us that the formula $X^{-1} = X^\dagger/\|X\|^2$ holds for any Clifford product $X = a_1 a_2 \cdots a_r \in C(V)$ of non-zero vectors $a_1, \ldots, a_r \in V$. However this is not true for a general non-zero element of $C(V)$. For example, since $(e_1 e_2 + e_3 e_4)(e_1 e_2 - e_3 e_4) = 0$, neither $e_1 e_2 + e_3 e_4$ nor $e_1 e_2 - e_3 e_4$ is invertible.

§3. Gram-Schmidt orthogonalization. The vectors $u_1, \ldots, u_r$ in Lemma 2 are given by a simple formula.

Proposition 3. (i) $u_k = (v_1 \wedge \cdots \wedge v_{k-1})^{-1}(v_1 \wedge \cdots \wedge v_k) \qquad (k = 2, \ldots, r)$.
(ii) *If one puts* $\tilde{e}_k := u_k/\|u_k\|$ $(k = 1, \ldots, r)$, *then* $\{\tilde{e}_1 \ldots, \tilde{e}_r\}$ *is nothing but the orthonormal family obtained by the Gram-Schmidt orthogonalization from* $\{v_1, \ldots, v_r\}$.

Proof. Similarly to (4), we have for $k = 2, \ldots, r$,

$$\begin{aligned} v_1 \wedge \cdots \wedge v_{k-1} \wedge v_k = u_1 \wedge \cdots \wedge u_{k-1} \wedge u_k &= (u_1 \cdots u_{k-1})u_k \\ &= (v_1 \wedge \cdots \wedge v_{k-1})u_k, \end{aligned}$$

whence (i) follows. The assertion (ii) follows from (3). □

§4. Reflection and projection. We denote by $V(v_1, \ldots, v_r)$ the subspace of V spanned by the vectors $v_1, \ldots, v_r$, and by $V(v_1, \ldots, v_r)^\perp$ its orthogonal complement in V. We note that any vector $x \in V$ is decomposed uniquely as $x = x_{\|} + x_\perp$ with $x_{\|} \in V(v_1, \ldots, v_r)$ and $x_\perp \in V(v_1, \ldots, v_r)^\perp$. As in Proposition 1, we put $A := v_1 \wedge v_2 \wedge \cdots \wedge v_r$.

Proposition 4. *One has* $(-1)^r AxA^{-1} = -x_{\|} + x_{\perp}$. *In other words, the linear map* $R_A : V \ni x \mapsto (-1)^r AxA^{-1} \in V$ *gives the reflection with respect to the space* $V(v_1, \dots, v_r)^{\perp}$.

For the case $r = 1$, Proposition 4 means a well-known description of the reflection with respect to a hyperplane.

Proof. We recall again the vectors $u_1, \dots, u_r$ in Lemma 2. If $x = x_{\perp} \in V(v_1, \dots, v_r)^{\perp}$, then we have $xu_k = -u_k x \ \ (k = 1, \dots, r)$, so that

$$\begin{aligned} xA &= xu_1u_2\cdots u_r = -u_1xu_2\cdots u_r = (-1)^2u_1u_2xu_3\cdots u_r = \dots \\ &= (-1)^r u_1u_2\cdots u_rx = (-1)^r Ax. \end{aligned}$$

Thus we obtain $R_A(x) = (-1)^r AxA^{-1} = x$ in the case $x = x_{\perp}$. On the other hand, if $x = x_{\|} \in V(v_1, \dots, v_r)$ and $x \neq 0$, we can take vectors $y_1, \dots, y_{r-1} \in V(v_1, \dots, v_r)$ for which $\{x,\, y_1,\, \dots, y_{r-1}\}$ forms an orthogonal basis of $V(v_1, \dots, v_r)$. Then we have

$$v_1 \wedge v_2 \wedge \cdots \wedge v_r = c_0\, x \wedge y_1 \wedge \dots y_{r-1} = c_0\, xy_1\cdots y_{r-1}$$

for some scalar $c_0 \in \mathbb{R}$. Since $xy_k = -y_kx \ \ (k = 1, \dots, r-1)$, we observe

$$\begin{aligned} xA &= x(c_0\, xy_1y_2\cdots y_{r-1}) = -c_0\, xy_1xy_2\cdots y_{r-1} = (-1)^2c_0\, xy_1y_2xy_3\cdots y_{r-1} = \dots \\ &= (-1)^{r-1}c_0\, xy_1y_2\cdots y_{r-1}x = (-1)^{r-1}Ax. \end{aligned}$$

Thus we obtain $R_A(x) = (-1)^r AxA^{-1} = -x$ in the case $x = x_{\|} \neq 0$. But this is valid also for $x = 0$. Therefore, for general $x = x_{\|} + x_{\perp} \in V$, we have $R_A(x) = R_A(x_{\|}) + R_A(x_{\perp}) = -x_{\|} + x_{\perp}$. □

Since the map $C(V) \ni X \mapsto AXA^{-1} \in C(V)$ is an algebra automorphism on $C(V)$, we have for vectors $x_1, x_2, \dots, x_s \in V$,

$$Ax_1A^{-1} \wedge Ax_2A^{-1} \wedge \cdots \wedge Ax_sA^{-1} = A(x_1 \wedge x_2 \wedge \cdots \wedge x_s)A^{-1}.$$

Thus, the reflection of the s-vector $x_1 \wedge \cdots \wedge x_s$ is given by the following formula:

Lemma 5. $R_A(x_1) \wedge R_A(x_2) \wedge \cdots \wedge R_A(x_s) = (-1)^{rs} A(x_1 \wedge x_2 \wedge \cdots \wedge x_s)A^{-1}$.

From Proposition 4, we obtain the following results.

Proposition 6. *One has* $x_{\|} = (x - (-1)^r AxA^{-1})/2$ *and* $x_{\perp} = (x + (-1)^r AxA^{-1})/2$.

Proposition 7. (i) $V(v_1, \dots, v_r) = \{\, x \in V\,;\, xA = (-1)^{r-1}Ax \,\}$.
(ii) $V(v_1, \dots, v_r)^{\perp} = \{\, x \in V\,;\, xA = (-1)^r Ax \,\}$.

§5. Inner and outer product of a vector with an r-vector. For a vector $x \in V$ and an r-vector B, we define

$$x \rfloor B := (xB - (-1)^r Bx)/2, \qquad x \wedge B := (xB + (-1)^r Bx)/2.$$

As in the previous sections, we assume that the vectors $v_1, \dots, v_r$ are linearly independent, and that $x = x_{||} + x_{\perp}$ ($x_{||} \in V(v_1, \dots, v_r)$, $x_{\perp} \in V(v_1, \dots, v_r)^{\perp}$), $A = v_1 \wedge \cdots \wedge v_r = u_1 \cdots u_r$.

Proposition 8. (i) *One has*

$$\text{(5)} \qquad x \rfloor (v_1 \wedge \cdots \wedge v_r) = \sum_{i=1}^{r} (-1)^{i-1} (x|v_i)(v_1 \wedge \cdots \check{v}_i \cdots \wedge v_r) \in C^{r-1}(V),$$

$$\text{(6)} \qquad x \wedge (v_1 \wedge \cdots \wedge v_r) = x \wedge v_1 \wedge \cdots \wedge v_r \in C^{r+1}(V),$$

where $\check{v}_i$ in (5) means that the i-th vector is omitted from the outer product.
(ii) *One has $x_{||} = (x \rfloor A)A^{-1}$ and $x_{\perp} = (x \wedge A)A^{-1}$.*
(iii) *The subspaces are described as $V(v_1, \dots, v_r) = \{ x \in V \,;\, x \wedge A = 0 \}$ and $V(v_1, \dots, v_r)^{\perp} = \{ x \in V \,;\, x \rfloor A = 0 \}$.*

Proof. We observe that

$$\begin{aligned} xv_1v_2 \cdots v_r = {} & (xv_1 + v_1x)v_2 \cdots v_r \\ & - v_1(xv_2 + v_2x)v_3 \cdots v_r + (-1)^2 v_1v_2(xv_3 + v_3x)v_4 \cdots v_r + \dots \\ & + (-1)^{r-1} v_1 \cdots v_{r-1}(xv_r + v_rx) + (-1)^r v_1v_2 \cdots v_rx, \end{aligned}$$

which is rewritten as

$$\begin{aligned} & (xv_1v_2 \cdots v_r - (-1)^r v_1v_2 \cdots v_rx)/2 \\ & = (x|v_1)v_2 \cdots v_r - (x|v_2)v_1v_3 \cdots v_r + (-1)^2 (x|v_3)v_1v_2v_4 \cdots v_r + \dots \\ & \quad + (-1)^{r-1}(x|v_r)v_1 \cdots v_{r-1}, \end{aligned}$$

because $(xv_i + v_ix)/2 = (x|v_i)$ ($i = 1, \dots, r$). Replacing the indices $1, 2, \dots, r$ by $\sigma(1), \sigma(2), \dots, \sigma(r)$ for $\sigma \in \mathfrak{S}_r$, and summing up each term as in (1), we obtain (5).

We have $x_{\perp}A = (xA + (-1)^r Ax)/2 = x \wedge A$ by Proposition 6. On the other hand,

$$x_{\perp}A = x_{\perp}u_1u_2 \cdots u_r = x_{\perp} \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_r = x \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_r,$$

where the last equality follows from (3) and $x_{||} \in V(v_1, \dots, v_r)$. Thus we get (6).

The assertions (ii) and (iii) follow from Propositions 6 and 7 respectively. □

In view of (5) and (6), we call $x\rfloor B$ (resp. $x \wedge B$) the *inner* (resp. *outer*) *product* of x with B. By definition we have

$$xB = x\rfloor B + x \wedge B. \tag{7}$$

Note that the Clifford product is invertible if B is a non-zero simple r-vector, while the inner nor the outer product is not in general.

§6. Dual basis. A basis $\{w_1, \dots, w_r\}$ of the subspace $V(v_1, \dots, v_r)$ is said to be *dual* to $\{v_1, \dots, v_r\}$ if $(v_i|w_j) = \delta_{ij}$ $(1 \leq i, j \leq r)$, where δ_{ij} is Kronecker's delta.

Proposition 9. *The dual basis $\{w_1, \dots, w_r\}$ is given by the following formula:*

$$w_k = (-1)^{k-1}(v_1 \wedge \cdots \check{v}_k \cdots \wedge v_r)(v_1 \wedge \cdots \wedge v_r)^{-1}.$$

Proof. Since $w_k \in V(v_1, \dots, v_r)$, Proposition 8 (iii) together with (7) tells us that $w_k A = w_k \rfloor A$. On the other hand, we have by (5)

$$\begin{aligned} w_k \rfloor A &= \sum_{i=1}^{r} (-1)^{i-1}(w_k|v_i)(v_1 \wedge \cdots \check{v}_i \cdots \wedge v_r) \\ &= (-1)^{k-1} v_1 \wedge \cdots \check{v}_k \cdots \wedge v_r \end{aligned}$$

because $(w_k|v_i) = \delta_{ki}$. Therefore we have $w_k(v_1 \wedge \cdots \wedge v_r) = (-1)^{k-1} v_1 \wedge \cdots \check{v}_k \cdots \wedge v_r$, whence the formula follows. □

§7. Concluding comments. We have seen that various calculations and formulas are described quite simply by using the Clifford algebra if we just introduce a few of notions such as the outer product of vectors. But they are merely a small part of concepts which enrich the Clifford algebra (see [2]). For example, the notion of *Clifford analytic function* (or *monogenic function*) extends naturally the theory of complex analysis ([1], [2]). On the other hand, if one utilizes the Clifford algebra of signature $(3, 1)$ (*the spacetime algebra*), the formulas in special relativity and electromagnetism like the Maxwell equation are rewritten impressively ([1], [4]).

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